JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **168**, 596–604 (1992)

A Refinement of the Cauchy–Schwarz Inequality

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Submitted by R. P. Boas

Received March 22, 1991

DEDICATED TO PROFESSOR DR. J. L. BRENNER ON THE OCCASION OF HIS 80TH BIRTHDAY

We prove: If x_k and y_k $(k = 1, ..., n)$ are real numbers satisfying

$$0 = x_0 < x_1 \leqslant x_2/2 \leqslant \cdots \leqslant x_n/n \qquad \text{and} \qquad 0 < y_n \leqslant y_{n-1} \leqslant \cdots \leqslant y_1,$$

then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leqslant \sum_{k=1}^{n} y_k \sum_{k=1}^{n} (x_k^2 - \tfrac{1}{4} x_k x_{k-1}) y_k \tag{$*$}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, ..., n)$ and $y_1 = \cdots = y_n$. Inequality $(*)$ is valid, in particular, if the sequence (x_k) is positive and convex.

1. INTRODUCTION

The famous Cauchy–Schwarz inequality states:

If $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ are two n-tuples of real numbers, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leqslant \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 \tag{1.1}$$

with equality holding if and only if $\mathbf{x}$ and $\mathbf{y}$ are proportional. This result, which is also called the Cauchy–Schwarz–Buniakowski inequality or simply the Cauchy inequality, has evoked tremendous interest among many mathematicians and numerous extensions, variants, and inverses of (1.1) were published; see [1, 3, and references therein].

0022-247X/92 $5.00

It is well known that under additional assumptions inequalities can be sharpened. Remarkable refinements for all classical inequalities can be found in literature. In particular there exist interesting sharpenings of (1.1). We mention two of them:

In 1952 A. Ostrowski [4, p. 289] proved:

If $\mathbf{x}=(x_1, ..., x_n)$, $\mathbf{y}=(y_1, ..., y_n)$, and $\mathbf{z}=(z_1, ..., z_n)$ are n-tuples of real numbers such that $\mathbf{x}$ and $\mathbf{y}$ are not proportional and

$$\sum_{k=1}^{n} x_k z_k = 0 \qquad \text{and} \qquad \sum_{k=1}^{n} y_k z_k = 1,$$

then

$$\sum_{k=1}^{n} x_k^2 \Big/ \sum_{k=1}^{n} z_k^2 \leqslant \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 - \left(\sum_{k=1}^{n} x_k y_k\right)^2.$$

The following proposition was given by H. W. McLaughlin [2] in 1966: If $\mathbf{x}=(x_1, ..., x_{2n})$ and $\mathbf{y}=(y_1, ..., y_{2n})$ are $2n$-tuples of real numbers, then

$$\left[\sum_{k=1}^{n} (x_{2k} y_{2k-1} - x_{2k-1} y_{2k})\right]^2 \leqslant \sum_{k=1}^{2n} x_k^2 \sum_{k=1}^{2n} y_k^2 - \left(\sum_{k=1}^{2n} x_k y_x\right)^2.$$

Further refinements of (1.1) were published in the monographs [1, 3]. The aim of this paper is to present a sharpening of the Cauchy–Schwarz inequality written in the form

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leqslant \sum_{k=1}^{n} y_k \sum_{k=1}^{n} x_k^2 y_k.$$

In Section 3 we prove that the inequality

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leqslant \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \left(x_k^2 - \frac{1}{4} x_k x_{k-1}\right) y_k \tag{1.2}$$

is valid for all n-tuples $\mathbf{x}=(x_1, ..., x_n)$ and $\mathbf{y}=(y_1, ..., y_n)$ such that

$$0 = x_0 < x_1 \leqslant x_2/2 \leqslant \cdots \leqslant x_n/n$$

and

$$0 < y_n \leqslant y_{n-1} \leqslant \cdots \leqslant y_1.$$

The sign of equality holds in (1.2) if and only if $x_k = kx_1$ $(k=1, ..., n)$ and $y_1 = \cdots = y_n$.

The proof of this theorem is rather long. We start by formulating and establishing three lemmas which we need later.

2. Three Lemmas

Lemma 1. *If* $0 = x_0 < x_1 \leqslant x_2/2 \leqslant \cdots \leqslant x_n/n$, *then*

$$2 \sum_{k=1}^{n} x_k \leqslant (n+1)\, x_n \tag{2.1}$$

and

$$0 \leqslant \frac{3(n+1)^2}{4n} x_n^2 - \frac{2(n+1)}{n} x_n \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} \left(x_k^2 - \frac{1}{4} x_k x_{k-1} \right). \tag{2.2}$$

Equality holds in (2.2) if and only if $x_k = k x_1$ $(k = 1, ..., n)$.

Proof. We have for $k = 1, ..., n-1: (k+1)\, x_k \leqslant k x_{k+1}$. Adding yields

$$\begin{aligned} 2x_1 + \sum_{k=2}^{n-1} (k+1)\, x_k = \sum_{k=1}^{n-1} (k+1)\, x_k &\leqslant \sum_{k=1}^{n-1} k x_{k+1} \\ &= \sum_{k=2}^{n-1} (k-1)\, x_k + (n-1)\, x_n \end{aligned}$$

which implies

$$2x_1 + 2 \sum_{k=2}^{n-1} x_k + 2x_n = 2 \sum_{k=1}^{n} x_k \leqslant (n+1)\, x_n.$$

We prove (2.2) by induction on n. For $n = 1$ the assertion is true. Next we assume inequality (2.2) holds with $n-1$ instead of n and we define for $n \geqslant 2$ and $x_n \geqslant (n/(n-1))\, x_{n-1}$:

$$f(x_n) = \frac{3(n+1)^2}{4n} x_n^2 - \frac{2(n+1)}{n} x_n \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} \left(x_k^2 - \frac{1}{4} x_k x_{k-1} \right).$$

Differentiation leads to

$$f'(x_n) = \frac{3n^2 + 6n - 1}{2n} x_n - \frac{2(n+1)}{n} \sum_{k=1}^{n} x_k - \frac{1}{4} x_{n-1}.$$

From (2.1) and $x_n \geqslant (n/(n-1))\, x_{n-1}$ we get

$$f'(x_n) \geqslant \frac{2n^2 + 3n - 5}{4n} x_n > 0.$$

Thus we obtain

$$f(x_n) \geqslant f\left(\frac{n}{n-1} x_{n-1} \right) \tag{2.3}$$

with equality holding if and only if $x_n=(n/(n-1))\,x_{n-1}$. A short calculation reveals that

$$f\left(\frac{n}{n-1}x_{n-1}\right)=\left[\frac{3n^2}{4(n-1)}x_{n-1}^2-\frac{2n}{n-1}x_{n-1}\sum_{k=1}^{n-1}x_k+\sum_{k=1}^{n-1}\left(x_k^2-\frac{1}{4}x_kx_{k-1}\right)\right]$$
$$+\frac{x_{n-1}}{n-1}\left[nx_{n-1}-2\sum_{k=1}^{n-1}x_k\right]. \tag{2.4}$$

From the induction hypothesis and from (2.1) we conclude that both terms in the square brackets of (2.4) are non-negative, so that (2.3) and (2.4) imply

$$f(x_n)\geqslant 0.$$

If $f(x_n)=0$, then we obtain $x_n=(n/(n-1))\,x_{n-1}$ and since the expression in the first square brackets of (2.4) must vanish as well we conclude from the induction hypothesis that

$$x_k=kx_1 \qquad (k=1,\ldots,n-1).$$

Because of $x_n=(n/(n-1))\,x_{n-1}=nx_1$ we have

$$x_k=kx_1 \qquad (k=1,\ldots,n)$$

which completes the proof of Lemma 1. ∎

The results of Lemma 1 are important to establish

Lemma 2. *If* $0=x_0<x_1\leqslant x_2/2\leqslant\cdots\leqslant x_{n+1}/(n+1)$, *then*

$$0\leqslant nx_{n+1}^2-\frac{n+1}{4}x_nx_{n+1}-2x_{n+1}\sum_{k=1}^{n}x_k+\sum_{k=1}^{n}\left(x_k^2-\frac{1}{4}x_kx_{k-1}\right)$$

with equality holding if and only if $x_k=kx_1$ $(k=1,\ldots,n+1)$.

Proof. We define

$$g(x_{n+1})=nx_{n+1}^2-\frac{n+1}{4}x_nx_{n+1}-2x_{n+1}\sum_{k=1}^{n}x_k+\sum_{k=1}^{n}\left(x_k^2-\frac{1}{4}x_kx_{k-1}\right).$$

Differentiation yields

$$g'(x_{n+1})=2nx_{n+1}-\frac{n+1}{4}x_n-2\sum_{k=1}^{n}x_k.$$

From (2.1) and the assumption $x_{n+1} \geqslant ((n+1)/n)\, x_n$ we conclude

$$g'(x_{n+1}) \geqslant \frac{3}{4}(n+1)\, x_n > 0$$

which implies

$$g(x_{n+1}) \geqslant g\left(\frac{n+1}{n}\, x_n\right) \tag{2.5}$$

with equality holding if and only if $x_{n+1} = ((n+1)/n)\, x_n$. Because of

$$g\left(\frac{n+1}{n}\, x_n\right) = \frac{3(n+1)^2}{4n}\, x_n^2 - \frac{2(n+1)}{n}\, x_n \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} \left(x_k^2 - \frac{1}{4}\, x_k x_{k-1}\right)$$

we obtain from (2.2) and (2.5)

$$g(x_{n+1}) \geqslant 0.$$

If $g(x_{n+1}) = 0$, then we have $x_{n+1} = ((n+1)/n)\, x_n$ and since $g(((n+1)/n)\, x_n) = 0$ we conclude from Lemma 1 that $x_k = kx_1$ $(k = 1, ..., n)$ which implies that $x_k = kx_1$ $(k = 1, ..., n+1)$. ∎

Now we prove the validity of inequality (1.2) for the special case $y_1 = \cdots = y_n = 1$.

LEMMA 3. *If* $0 = x_0 < x_1 \leqslant x_2/2 \leqslant \cdots \leqslant x_n/n$, *then*

$$\left(\sum_{k=1}^{n} x_k\right)^2 \leqslant n \sum_{k=1}^{n} \left(x_k^2 - \frac{1}{4}\, x_k x_{k-1}\right) \tag{2.6}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, ..., n)$.

Proof. We use induction on n. For $n = 1$ the assertion is obviously true. Next we assume that the proposition holds for $n \geqslant 1$. Then we conclude from the induction hypothesis and from Lemma 2 that

$$\begin{aligned}
\left(\sum_{k=1}^{n+1} x_k\right)^2 &= \left(\sum_{k=1}^{n} x_k\right)^2 + 2x_{n+1} \sum_{k=1}^{n} x_k + x_{n+1}^2 \\
&\leqslant n \sum_{k=1}^{n} \left(x_k^2 - \frac{1}{4}\, x_k x_{k-1}\right) + 2x_{n+1} \sum_{k=1}^{n} x_k + x_{n+1}^2 \\
&\leqslant (n+1) \sum_{k=1}^{n+1} \left(x_k^2 - \frac{1}{4}\, x_k x_{k-1}\right).
\end{aligned}$$

If $(\sum_{k=1}^{n+1} x_k)^2 = (n+1) \sum_{k=1}^{n+1} (x_k^2 - \frac{1}{4} x_k x_{k-1})$, then the sign of equality holds in particular in the last inequality and from Lemma 2 we conclude $x_k = kx_1$ $(k = 1, ..., n+1)$. ∎

3. The Main Result

Now we are in a position to prove the theorem stated in the Introduction.

THEOREM. *Let* $\mathbf{x}=(x_1, ..., x_n)$ *and* $\mathbf{y}=(y_1, ..., y_n)$ *be n-tuples of real numbers such that*

$$0=x_0<x_1 \leqslant x_2/2 \leqslant \cdots \leqslant x_n/n$$

and

$$0<y_n \leqslant y_{n-1} \leqslant \cdots \leqslant y_1.$$

Then we have

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leqslant \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \left(x_k^2-\frac{1}{4} x_k x_{k-1}\right) y_k \tag{3.1}$$

with equality holding if and only if $x_k=kx_1$ $(k=1, ..., n)$ *and* $y_1=\cdots=y_n$.

Proof. If $y_1=\cdots=y_n$, then inequalities (3.1) and (2.6) are equivalent. Hence we may assume $y_1>y_n$ $(n \geqslant 2)$ and we prove (3.1) with "$<$" instead of "$\leqslant$" by induction on n.

We define for $y_1>y_2$

$$h(y_1)=\sum_{k=1}^{2} y_k \sum_{k=1}^{2} \left(x_k^2-\frac{1}{4} x_k x_{k-1}\right) y_k-\left(\sum_{k=1}^{2} x_k y_k\right)^2.$$

Because of $2x_1 \leqslant x_2$ we conclude that

$$h'(y_1)=y_2\left(x_1^2+x_2^2-\frac{9}{4} x_1 x_2\right) \geqslant \frac{1}{4} y_2 x_1 x_2>0,$$

and from (2.6) with $n=2$ we obtain

$$h(y_1)>h(y_2)=y_2^2\left[2 \sum_{k=1}^{2} \left(x_k^2-\frac{1}{4} x_k x_{k-1}\right)-\left(\sum_{k=1}^{2} x_k\right)^2\right] \geqslant 0.$$

Next we suppose that the assertion is true for all natural numbers less than n. We define

$$\varphi(y_1, ..., y_n)=\sum_{k=1}^{n} y_k \sum_{k=1}^{n} \left(x_k^2-\frac{1}{4} x_k x_{k-1}\right) y_k-\left(\sum_{k=1}^{n} x_k y_k\right)^2$$

and for $y \geqslant y_{q+1}$

$$\varphi_q(y)=\varphi(y, ..., y, y_{q+1}, ..., y_n) \qquad (q=1, ..., n-1).$$

We show that φ_q is strictly increasing on $[y_{q+1}, \infty)$.

We differentiate φ_q twice and apply inequality (2.6); then we have

$$\frac{1}{2}\varphi_q''(y)=q\sum_{k=1}^{q}\left(x_k^2-\frac{1}{4}x_k x_{k-1}\right)-\left(\sum_{k=1}^{q}x_k\right)^2\geqslant 0$$

which implies

$$\begin{aligned}\frac{1}{q}\varphi_q'(y)\geqslant\frac{1}{q}\varphi_q'(y_{k+1})&\\ =2y_{k+1}&\left\{\sum_{k=1}^{q}\left(x_k^2-\frac{1}{4}x_k x_{k-1}\right)-\frac{1}{q}\left(\sum_{k=1}^{q}x_k\right)^2\right\}\\ &+\left[\frac{1}{q}\sum_{k=q+1}^{n}y_k\sum_{k=1}^{q}\left(x_k^2-\frac{1}{4}x_k x_{k-1}\right)\right.\\ &\left.+\sum_{k=q+1}^{n}\left(x_k^2-\frac{1}{4}x_k x_{k-1}\right)y_k-\frac{2}{q}\sum_{k=1}^{q}x_k\sum_{k=q+1}^{n}x_k y_k\right].\end{aligned}$$

From (2.6) we conclude that the difference in curled braces is non-negative. If we denote the expression in square brackets by Δ, then we obtain from (2.6) and the induction hypothesis

$$\begin{aligned}\Delta\sum_{k=q+1}^{n}y_k\geqslant&\left(\frac{1}{q}\sum_{k=q+1}^{n}y_k\sum_{k=1}^{q}x_k\right)^2+\left(\sum_{k=q+1}^{n}x_k y_k\right)^2\\ &-\frac{2}{q}\sum_{k=q+1}^{n}y_k\sum_{k=1}^{q}x_k\sum_{k=q+1}^{n}x_k y_k\\ \geqslant&\, 0.\end{aligned}$$

If (for a contradiction) $\Delta=0$, then we could conclude from Lemma 3 and the induction hypothesis that

$$x_k=kx_1 \qquad (k=1, ..., n) \tag{3.2}$$

and

$$y_{q+1}=\cdots=y_n; \tag{3.3}$$

further we would have

$$\frac{1}{q}\sum_{k=q+1}^{n}y_k\sum_{k=1}^{q}x_k=\sum_{k=q+1}^{n}x_k y_k. \tag{3.4}$$

From (3.2) and (3.3) it would follow that (3.4) was equivalent to $q=n$, which would contradict the assumption $q \leqslant n-1$.

Hence we have $\Delta > 0$ which leads to

$$\varphi_q'(y) > 0 \qquad \text{for} \quad y \geqslant y_{q+1}.$$

Because

$$\varphi_r(y_{r+1}) = \varphi_{r+1}(y_{r+1}) \qquad (r = 1, ..., n-2)$$

we get

$$\begin{aligned} \varphi(y_1, ..., y_n) &= \varphi_1(y_1) \geqslant \varphi_1(y_2) = \varphi_2(y_2) \geqslant \cdots \\ &\geqslant \varphi_{n-2}(y_{n-1}) = \varphi_{n-1}(y_{n-1}) \geqslant \varphi_{n-1}(y_n) \qquad (3.5) \\ &= \varphi(y_n, ..., y_n) = y_n^2 \left[n \sum_{k=1}^{n} \left(x_k^2 - \frac{1}{4} x_k x_{k-1} \right) - \left(\sum_{k=1}^{n} x_k \right)^2 \right] \end{aligned}$$

and from (2.6) we obtain

$$\varphi(y_1, ..., y_n) \geqslant 0.$$

Since φ_q is strictly increasing on $[y_{q+1}, \infty)$ and because $y_i > y_{i+1}$ for a number $i \in \{1, ..., n-1\}$ we conclude that at least one of the inequalities in (3.5) is strict. Thus we get

$$\varphi(y_1, ..., y_n) > 0$$

which we had to prove. ∎

Remark. The theorem is in particular true if we assume that the sequence (x_k) is positive and convex; this means if $x_0 = 0$ and

$$2x_k \leqslant x_{k-1} + x_{k+1} \qquad (k = 1, ..., n-1).$$

Indeed, since $x_1 \leqslant x_2/2$ and

$$(k-1)\left[\frac{x_k}{k} - \frac{x_{k-1}}{k-1}\right] \leqslant (k+1)\left[\frac{x_{k+1}}{k+1} - \frac{x_k}{k}\right] \qquad (k = 2, ..., n-1)$$

it follows by induction that (x_k/k) is increasing.

References

1. P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, "Means and Their Inequalities," Reidel, Dordrecht, 1988.
2. H. W. McLaughlin, "Inequalities Complementary to the Cauchy–Schwarz Inequality for Finite Sums of Quaternions; Refinements of the Cauchy-Schwarz Inequality for Finite Sums of Real Numbers; Inequalities Concercing Ratios and Differences of Generalized Means of Different Order," University of Maryland, Technical Note BN-454, 1966.
3. D. S. Mitrinović, "Analytic Inequalities," Springer-Verlag, New York, 1970.
4. A. Ostrowski, "Vorlesungen über Differential- und Integralrechnung," Vol. 2, Birkhäuser, Basel, 1951.